BIPUNCTUAL COORDINATES.

By F. Franklin, *Fellow of the Johns Hopkins University.*

The expressions for the bilinear coordinates of a point in terms of its trilinear coordinates contain a common denominator which is a linear function of the latter. Whenever, therefore, the trilinear coordinates of a point are such as to make this function equal to zero, its bilinear coordinates are infinite; nor are they infinite under any other supposition: and hence the equation formed by putting this common denominator equal to zero is called the equation of the infinitely distant straight line.

When we examine the corresponding expressions for the bilinear coordinates of a *line*, we do not arrive at any corresponding geometrical idea. These expressions, too, have a common denominator of the first degree; but the equation obtained by putting this denominator equal to zero represents simply the origin of coordinates, a point of no geometrical importance.

I propose to construct a system of coordinates* in which the infinitely distant point shall hold a position similar to that held by the infinitely distant straight line in the bilinear system. In the bilinear system we begin by referring a point to two fixed lines by means of coordinates; we find that the equation of a straight line is of the first degree in the coordinates of its points; and we then define the coordinates of a straight line in such a way that they will be represented by the coefficients of the point-coordinates in its equation when put into a certain form. In the system proposed, a line will be referred to two fixed points by means of coordinates; we shall find that the equation of a point is of the first degree in the coordinates of its lines; and we shall define the coordinates of a point in such a way that they will be represented by the coefficients of the line-coordinates in its equation when put into a certain form, precisely like the corresponding form of the equation of a line in the bilinear system.

*When I had written the greater part of this paper, I accidentally discovered, through one of the notes appended to Salmon's "Conic Sections," that the system of *tangential* coordinates here presented had been used before; where, or to what extent, I do not know.

Plate II.

Fig. 1

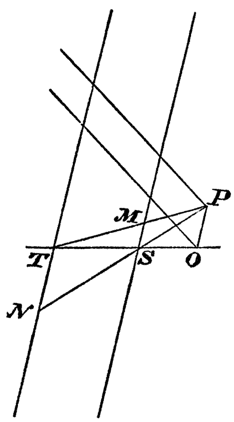


Fig. 2

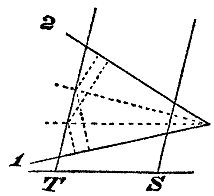


Fig. 3

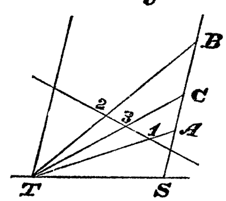


Fig. 4

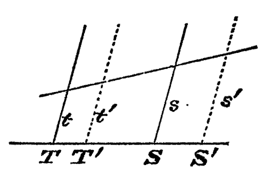


Fig. 5

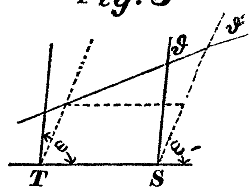


Fig. 6

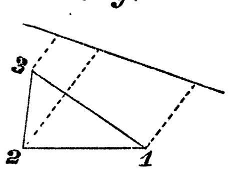


Fig. 7

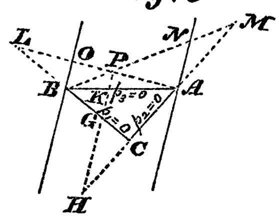


Fig. 8

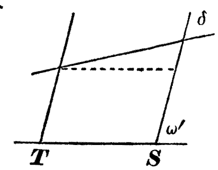


Fig 9

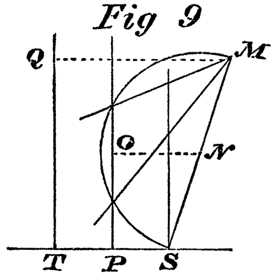
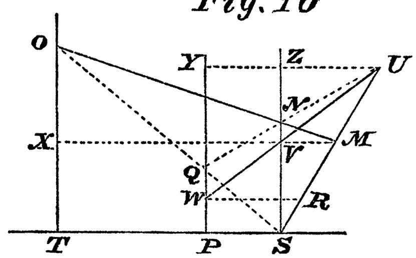


Fig. 10



Franklin, *Bipunctual Coordinates.*

Let the points S, T, (Fig. 1) to which we are to refer all lines, be called the *initials*, and the straight line joining them the *base.*

The coordinates s, t, of a line are its distances from the fixed points S, T, measured in a fixed direction—the same for both initials—which will be called the direction of reference; the lines passing through S and T in the direction of reference may be called the lines of reference.

The equation of any point O on the base is obviously $\frac{s}{t} = \frac{SO}{TO} = c$. Take any point P not on the base. Draw PO parallel to the lines of reference, and let $OP = d$. Then for any line of P, we have $\frac{s - d}{t - d} = c$; this, then, is the equation of P. It is an equation of the first degree; hence the equation of every point is of the first degree.

Conversely, every equation of the first degree is the equation of a point. For the equation above obtained can be written $s = ct + (1 - c)\,d$; and any equation of the first degree $As + Bt + C = 0$ can be written $s = -\frac{B}{A}\,t - \frac{C}{A}$; so that, to obtain the point represented by $As + Bt + C = 0$, we have only to take a point O on the base, such that $\frac{SO}{TO} = -\frac{B}{A}$, and from O to lay off, in the direction of reference, a distance $-\dfrac{\frac{C}{A}}{1 + \frac{B}{A}}$ or, $-\dfrac{C}{A + B}$.

There must, of course, be an understanding as to signs. When s or t is measured upward from the initial, it will be regarded as positive; when downward, negative. When the distance SO or TO is measured from the initial in the direction TS, it will be regarded as positive; when in the direction ST, negative. Thus for all points between S and T the ratio $\frac{SO}{TO}$ will be negative; for all points of the base outside of $\overline{ST}$ it will be positive. The above equations of points are in accord with this convention as to signs.

I will venture to make a slight innovation in mathematical language. I have spoken above of the lines of or on a point, meaning the lines passing through a point; for it seems to me that a closer analogy in our language respecting the point and the straight line would tend to facilitate both the comprehension and the remembrance of their geometrical analogies. For this reason, I propose to call the straight line joining two points their *junction;*

for the word *intersection* is used to designate the common point of two straight lines, and we ought to have a corresponding word, *junction*, to designate the common line of two points. And as the word *intersection* is applied to the common points of curves in general, so the word *junction* should be applied to the common lines of curves in general. Thus such expressions as "the tangents drawn from a point to a curve," "the common tangents of two curves," &c., would be replaced by the expressions "the junctions of a point with a curve," "the junctions of two curves," &c.

For the intersection of two lines s_1, t_1, and s_2, t_2, we have the equations $As + Bt + C = 0$, $As_1 + Bt_1 + C = 0$, $As_2 + Bt_2 + C = 0$, whence, eliminating A, B and C, we have for the equation of the point,

$$\begin{vmatrix} s, & t, & 1 \\ s_1, & t_1, & 1 \\ s_2, & t_2, & 1 \end{vmatrix} = 0,$$

If we take as the two lines the junctions of the point with the initials, we have (Fig. 1) $s_1 = SM = a$, $t_1 = 0$; $s_2 = 0$, $t_2 = TN = b$; and the equation of the point becomes

$$\begin{vmatrix} s, & t, & 1 \\ a, & 0, & 1 \\ 0, & b, & 1 \end{vmatrix} = ab - bs - at = 0,$$

or $\frac{s}{a} + \frac{t}{b} - 1 = 0$. If we put $-\frac{1}{a} = p$, $-\frac{1}{b} = q$, this equation becomes $ps + qt + 1 = 0$. Let us call p and q the coordinates of the point P; then we may regard our equation as expressing either the condition that a line s, t, should pass through a fixed point p, q, or the condition that a point p, q, should lie on a fixed line s, t. Thus the form of the combined equation of point and line is precisely the same as in the bilinear system; and we can save time and space by adopting at once such of the formulæ obtained for the bilinear system as depend simply upon the form of this equation. Thus we have:

For any line on the intersection of two lines s_1, t_1; s_2, t_2;

$$s = \frac{s_1 + \lambda s_2}{1 + \lambda}, \quad t = \frac{t_1 + \lambda t_2}{1 + \lambda}.$$

Hence $\lambda = \frac{s - s_1}{s_2 - s} = \frac{t - t_1}{t_2 - t}$.

For any point on the junction of two points* p_1, q_1; p_2, q_2;

$$p = \frac{p_1 + \lambda p_2}{1 + \lambda}, \quad q = \frac{q_1 + \lambda q_2}{1 + \lambda}.$$

Hence $\lambda = \frac{p - p_1}{p_2 - p} = \frac{q - q_1}{q_2 - q}$.

*It is hardly necessary to mention that the equation of the junction of two points p_1, q_1; p_2, q_2; is

$$\begin{vmatrix} p, & q, & 1 \\ p_1, & q_1, & 1 \\ p_2, & q_2, & 1 \end{vmatrix} = 0.$$

In the case of the line we see at once (Fig. 2) that λ is *proportional* to the distance-ratio of the movable line from the two fixed lines; *i. e.*, the ratio of the perpendiculars dropped upon the two fixed lines from any point of the movable line.

In the case of the point, the meaning of λ is not quite so obvious. Representing by α, β, the distances SA, SB, (Fig. 3) cut off on the S line of reference by the junctions of 1 and 2 respectively with the initial T, and by ρ the corresponding distance SC for any third point 3 of the line 12, we have $p_1 = -\frac{1}{\alpha}$, $p_2 = -\frac{1}{\beta}$, $p = -\frac{1}{\rho}$; so that

$$\lambda = \frac{p - p_1}{p_2 - p} = \frac{\dfrac{1}{\alpha} - \dfrac{1}{\rho}}{\dfrac{1}{\rho} - \dfrac{1}{\beta}} = \frac{\rho - \alpha}{\beta - \rho} \cdot \frac{\beta}{\alpha} = \frac{\beta}{\alpha} \cdot \frac{AC}{CB}.$$

Now, wherever the point 3 be taken, $\frac{AC}{CB}$ is equal to $\frac{(13)}{(32)}$ multiplied by a constant; so that λ, which is equal to $\frac{AC}{CB}$ multiplied by a constant, is *proportional* to the distance-ratio of the movable point from the two fixed points.

We have thus seen that the parameter λ has the same geometrical significance as the corresponding parameter in the bilinear system; it is needless, therefore, to state the theorems respecting the equations of straight lines passing through a point and of points lying on a straight line, and respecting their anharmonic ratio; the theorems and formulæ for the bipunctual system will be the same as those for the bilinear system, and we can make use of those obtained for the latter whenever we have occasion for them.

I shall consider only two metrical problems relating to the point and straight line; and these because they will be necessary in the transformation of coordinates; and here I shall introduce two words which may be convenient. The distance from a point to a straight line, measured in the direction of reference, will be called the *departure* of the point from the line or of the line from the point; the distance measured on a line parallel to the base will be called the *remove* of the point from the line or of the line from the point. Both these distances will be regarded as positive when the *line is on the positive side of the point.*

1. Required the departure of a line whose coordinates are given from a point whose equation is given.

Let s', t', be the coordinates of the line, and $as + bt + c = 0$ the equation of the point; represent the required departure by δ. The coordinates of the line passing through the given point and parallel to the given line are $s' - \delta$, $t' - \delta$, so that we have

$$a\,(s' - \delta) + b\,(t' - \delta) + c = 0\,; \text{ whence } \delta = \frac{as' + bt' + c}{a + b}$$

2. Required the departure of a point whose coordinates are given from a line whose equation is given.

Let p', q', be the coordinates of the point, and $ap + bq + c = 0$ the equation of the line; then the coordinates of the line are $\frac{a}{c}$ and $\frac{b}{c}$, and the equation of the point is $p's + q't + 1 = 0$: we have, therefore, from the previous case,

$$\delta = \frac{p'\frac{a}{c} + q'\frac{b}{c} + 1}{p' + q'} = \frac{ap' + bq' + c}{c\,(p' + q')}$$

TRANSFORMATION OF COORDINATES.

I. As long as the direction of reference remains fixed, the only change that can be made in a system of bipunctual coordinates is an alteration in the position of the initials, and any such alteration can be effected by first moving them along the base and then moving them in the direction of reference, by which last process the base itself is moved.

First, let the initials move along the base. Represent by L the distance ST (Fig. 4) between the old initials, by l the distance $S'T'$ between the new initials, by m and n the removes of the new reference-line at S' from S and T respectively, and by m' and n' the corresponding removes of the new reference-line at T'. These quantities are connected by the equations $n - m = n' - m' = L$, $m - m' = n - n' = l$. We have

$$s = s' - \frac{m}{l}\,(s' - t') = \frac{mt' - m's'}{l}\,, \quad t = t' - \frac{n'}{l}\,(s' - t') = \frac{nt' - n's'}{l}\,.$$

Secondly, let the initials move in the direction of reference. Representing by a and b the departures of the new base from the old initials, we have obviously $s = s' + a$, $t = t' + b$.

II. When the direction of reference is altered, we have, (Fig. 5)

$$\frac{s'}{s} = \frac{t'}{t} = \frac{\sin \vartheta}{\sin \vartheta'} = \frac{\sin\,(\vartheta' + \omega - \omega')}{\sin \vartheta'} = \cos\,(\omega - \omega') + \frac{\sin\,(\omega - \omega')}{\tan \vartheta'}$$

Now $$\frac{s'-t'}{L}=\frac{\sin(\omega'-\vartheta')}{\sin\vartheta'}=-\cos\omega'+\frac{\sin\omega'}{\tan\vartheta'},\quad\text{whence}$$

$$\frac{1}{\tan\vartheta'}=\frac{s'-t'+L\cos\omega'}{L\sin\omega'},\quad\text{so that we have}$$

$$\frac{s'}{s}=\frac{t'}{t}=\frac{L\left[\sin\omega'\cos(\omega-\omega')+\cos\omega'\sin(\omega-\omega')\right]+(s'-t')\sin(\omega-\omega')}{L\sin\omega'}$$

$$=\frac{L\sin\omega+(s'-t')\sin(\omega-\omega')}{L\sin\omega'},$$

or, $$s=s'\frac{L\sin\omega'}{L\sin\omega+(s'-t')\sin(\omega-\omega')},\quad t=t'\frac{L\sin\omega'}{L\sin\omega+(s'-t')\sin(\omega-\omega')}.$$

If the original system is rectangular and $L=1$, these equations become

$$s=\frac{s'}{\operatorname{cosec}\omega'+(s'-t')\cot\omega'},\qquad t=\frac{t'}{\operatorname{cosec}\omega'+(s'-t')\cot\omega'}.$$

The formulæ for point-coordinates can be immediately obtained from those for line-coordinates by observing the effect of the transformation of the latter upon the equation $ps+qt+1=0$.

TRIPUNCTUAL COORDINATES.

Just as the non-homogeneous equations of the bilinear system are replaced by homogeneous equations when we employ three lines of reference instead of two, the equations of the bipunctual system are replaced by homogeneous equations when we employ three points of reference.

We may define tripunctual coordinates as follows:

The coordinates of a line are three numbers which are to each other as the departures of the line from the three vertices of the triangle of reference, multiplied each by an arbitrary constant.

The coordinates of a point are three numbers which are to each other as the departures of the point from the three sides of the triangle of reference, multiplied each by an arbitrary constant.

Let us designate by s_1, s_2, s_3; p_1, p_2, p_3, the coordinates of line and point, respectively; by m_1, m_2, m_3, the departures of a line from the vertices 1, 2, 3, of the triangle of reference; and by n_1, n_2, n_3, those of a point from the sides 1, 2, 3; then we have

$$\mu s_1=\phi_1 m_1,\quad \mu s_2=\phi_2 m_2,\quad \mu s_3=\phi_3 m_3;\quad \nu p_1=\psi_1 n_1,\quad \nu p_2=\psi_2 n_2,\quad \nu p_3=\psi_3 n_3.$$

These tripunctual coordinates are proportional to trilinear coordinates; so that from any equation in trilinear coordinates we can obtain an equation

in tripunctnal coordinates by replacing $u_1, u_2, u_3, x_1, x_2, x_3$, by $s_1, s_2, s_3, p_1, p_2, p_3$. For, in the first place, representing by M_1, M_2, M_3, the *perpendicular* distances of a straight line from the vertices of the triangle of reference, we have $m_1 : m_2 : m_3 :: M_1 : M_2 : M_3$; and therefore if $\sigma u_i = \lambda_i M_i$ $(i = 1, 2, 3)$, we have only to take $\phi_i = k\lambda_i$ in order to have $s_1 : s_2 : s_3 :: u_1 : u_2 : u_3$. And secondly, representing by N_1, N_2, N_3, the *perpendicular* distances of a point from the sides of the triangle of reference, we have $N_1 = \alpha n_1$, $N_2 = \beta n_2$, $N_3 = \gamma n_3$, where α, β, and γ are constants (viz., the sines of the angles made by the sides of the triangle with a line drawn in the direction of reference); so that, if $\rho x_i = \varkappa_i N_i$, we have $\rho x_1 = \alpha \varkappa_1 n_1$, $\rho x_2 = \beta \varkappa_2 n_2$, $\rho x_3 = \gamma \varkappa_3 n_3$, and we have only to take $\psi_1 = k\alpha\varkappa_1$, $\psi_2 = k\beta\varkappa_2$, $\psi_3 = k\gamma\varkappa_3$, in order to have $p_1 : p_2 : p_3 :: x_1 : x_2 : x_3$.

Let us now obtain the equations connecting tripunctual with bipunctual coordinates.

Let the equations of the vertices of the triangle of reference be

$$\left.\begin{aligned} a_1 s + b_1 t + c_1 &= 0 \\ a_2 s + b_2 t + c_2 &= 0 \\ a_3 s + b_3 t + c_3 &= 0 \end{aligned}\right\}$$ where the determinant r, of the coefficients, is not zero.

Let the equations of the sides of the triangle of reference be

$$\left.\begin{aligned} A_1 p + B_1 q + C_1 &= 0 \\ A_2 p + B_2 q + C_2 &= 0 \\ A_3 p + B_3 q + C_3 &= 0 \end{aligned}\right\}$$ where the determinant R, of the coefficients, is not zero.

The coefficients $A_1 \ldots . C_3$ are equal (or at least proportional) to the corresponding minors in the determinant of the coefficients $a_1 \ldots . c_3$. We have (page 151)

for any line s, t,

$$m_1 = \frac{a_1 s + b_1 t + c_1}{a_1 + b_1},$$

$$m_2 = \frac{a_2 s + b_2 t + c_2}{a_2 + b_2},$$

$$m_3 = \frac{a_3 s + b_3 t + c_3}{a_3 + b_3}.$$

Taking $\phi_1 = a_1 + b_1$, $\phi_2 = a_2 + b_2$, $\phi_3 = a_3 + b_3$, we have $s_1 : s_2 : s_3 :: a_1 s + b_1 t + c_1$

$: a_2 s + b_2 t + c_2 : a_3 s + b_3 t + c_3$,

or, and, solving for s and t,

$$\mu s_1 = a_1 s + b_1 t + c_1 \qquad s = \frac{A_1 s_1 + A_2 s_2 + A_3 s_3}{C_1 s_1 + C_2 s_2 + C_3 s_3}$$

$$\mu s_2 = a_2 s + b_2 t + c_2$$

$$\mu s_3 = a_3 s + b_3 t + c^3 \qquad t = \frac{B_1 s_1 + B_2 s_2 + B_3 s_3}{C_1 s_1 + C_2 s_2 + C_3 s_3}$$

for any point p, q,

$$n_1 = \frac{A_1 p + B_1 q + C_1}{C_1 (p + q)},$$

$$n_2 = \frac{A_2 p + B_2 q + C_2}{C_2 (p + q)},$$

$$n_3 = \frac{A_3 p + B_3 q + C_3}{C_3 (p + q)}.$$

Taking $\psi_1 = C_1$, $\psi_2 = C_2$, $\psi_3 = C_3$, we have $p_1 : p_2 : p_3 :: A_1 p + B_1 q + C_1$

$: A_2 p + B_2 q + C_2 : A_3 p + B_3 q + C_3$,

or, and, solving for p and q,

$$\nu p_1 = A_1 p + B_1 q + C_1 \qquad p = \frac{a_1 p_1 + a_2 p_2 + a_3 p_3}{c_1 p_1 + c_2 p_2 + c_3 p_3}$$

$$\nu p_2 = A_2 p + B_2 q + C_2$$

$$\nu p_3 = A_3 p + B_3 q + C_3 \qquad q = \frac{b_1 p_1 + b_2 p_2 + b_3 p_3}{c_1 p_1 + c_2 p_2 + c_3 p_3}$$

These equations are precisely the same as those connecting Cartesian with trilinear coordinates; we can, therefore, adopt at once the algebraical consequences of the equations. Thus we have $\mu\nu\,(p_1s_1+p_2s_2+p_3s_3)=r\,(ps+qt+1)$, and the combined equation of point and line is $p_1s_1+p_2s_2+p_3s_3=0$.

The bipunctual system may be regarded as a special case of the tripunctual system. Take, first, the coordinates of a line (Fig. 6); designate by s and t its departures from the vertices 1 and 2, by τ its departure from the vertex 3, and by ρ the departure of 3 from the line 12. We have $\mu s_1=\phi_1 s$, $\mu s_2=\phi_2 t$, $\mu s_3=\phi_3\tau$. Take $\phi_1=1$, $\phi_2=1$, $\phi_3=\frac{1}{\rho}$; then $\mu s_1=s$, $\mu s_2=t$, $\mu s_3=\frac{\tau}{\rho}$. Now let 3 move in the direction of reference to an infinite distance; the limit of the ratio $\frac{\tau}{\rho}$ is unity, and we have $s_1:s_2:s_3::s:t:1$.

Secondly, take the coordinates of a point. The tripunctual coordinates of the point P (Fig. 7) are given by $\nu p_1=\psi_1 n_1$, $\nu p_2=\psi_2 n_2$, $\nu p_3=\psi_3 n_3$, where $n_1=-PG$, $n_2=-PH$, $n_3=-PK$. Take $\psi_1=\frac{1}{AC}$, $\psi_2=\frac{1}{BC}$, $\psi_3=-1$, so that $\nu p_1=\frac{n_1}{AC}$, $\nu p_2=\frac{n_2}{BC}$ $\nu p_3=-n_3$. The coordinates of P in the bipunctual system having A and B for its initials (the direction of reference remaining unchanged) are $p=-\frac{1}{AN}$, $q=-\frac{1}{BO}$. We have $PG:OB::LG:LB$ and $OB:PK::AB:AK$, whence $PG=PK\frac{AB}{AK}\cdot\frac{LG}{LB}$, $PH:NA::MH:MA$ and $NA:PK::AB:BK$, whence $PH=PK\frac{AB}{BK}\cdot\frac{MH}{MA}$; therefore $\nu p_1=-\frac{PK}{LB}\cdot\frac{AB}{AK}\cdot\frac{LG}{AC}$, $\nu p_2=-\frac{PK}{MA}\cdot\frac{AB}{BK}\cdot\frac{MH}{BC}$, $\nu p_3=PK$. Now let C move in the direction of reference to an infinite distance; at the limit, the lines AC, BC, PG, are parallel, LB is OB, MA is NA, and we have $\frac{PK}{LB}=\frac{AK}{AB}$, $\frac{PK}{MA}=\frac{BK}{AB}$, so that $\nu p_1=-\frac{LG}{AC}$, $\nu p_2=-\frac{MH}{BC}$, $\nu p_3=PK$. But at the limit we have also $\frac{LG}{AC}=\frac{BK}{AB}=\frac{PK}{AN}$ and $\frac{MH}{BC}=\frac{AK}{AB}=\frac{PK}{BO}$; so that $\nu p_1=-\frac{PK}{AN}$, $\nu p_2=-\frac{PK}{BO}$, $\nu p_3=PK$; that is, $p_1:p_2:p_3::p:q:1$.

If, in the expressions for s and t (p. 154, end), we put the common denominator equal to zero, we obtain the equation of the locus of all lines whose coordinates are infinite; that is, of all infinitely distant lines and all lines drawn in the direction of reference; and since the equation is of the first degree, we must regard this locus as a point: *the infinitely distant point* we may call it for convenience. Regarded analytically, all lines whose coordinates are infinite, pass through the infinitely distant point (*i. e.*, the infinitely distant point proper to the direction of reference); and conversely. Moreover, we must regard *all* infinitely distant points as lying on *one* infinitely distant straight line; for the condition that a point should be infinitely distant is obviously $p = -q$, an equation of the first degree.

We have seen that tripunctual coordinates are perfectly interchangeable with trilinear coordinates; it is needless, therefore, to say anything about transformation from one system of tripunctual coordinates to another. The coefficients of substitution will have the same geometrical meaning as those for trilinear coordinates. The two systems are, in fact, practically identical; whatever can be proved for or with the one can be proved for or with the other; and it is only in their relations to the bipunctual and bilinear systems that the distinction between the tripunctual and the trilinear systems comes into play: tripunctual coordinates standing in the same relation to bipunctual coordinates as trilinear to bilinear.

Defined as anharmonic ratios, the bipunctual coordinates of a line correspond precisely to the bilinear coordinates of a point:

Designate by X the line from which x is measured. The number which represents the x of any point is the anharmonic ratio of four lines passing through the intersection of X with the infinitely distant line; namely, X, the infinitely distant line, the line passing through the point considered, and a fixed line whose position determines the unit of length.

Designate by S the point from which s is measured. The number which represents the s of any line is the anharmonic ratio of four points lying on the junction of S with the infinitely distant point; namely, S, the infinitely distant point, the point lying on the line considered, and a fixed point whose position determines the unit of length.

THE CONIC SECTIONS REFERRED TO BIPUNCTUAL COORDINATES.

Let us see how some of the leading features of curves of the second class, or conics, present themselves when these curves are investigated by means of bipunctual line-coordinates. If we take the infinitely distant point as one vertex of a polar triangle, the opposite side is a diameter of the conic; let us designate this diameter as the base-diameter, and the points lying on it as basics. Any basic being taken as a second vertex of the polar triangle, the third vertex will be another basic harmonically situated with respect to the first basic and the intersections of the diameter with the curve. Two such points will be called conjugate basics; their analogy with conjugate diameters may be set forth thus:

Conjugate basics are two points lying on the base-diameter, so situated that the junction of each with the infinitely distant point is the polar of the other.	Conjugate diameters are two lines passing through the centre, so situated that the intersection of each with the infinitely distant line is the pole of the other.
All conjugate basics are harmonically situated with respect to the basics lying on the junctions of the curve with the infinitely distant point.	All conjugate diameters are harmonically situated with respect to the diameters passing through the intersections of the curve with the infinitely distant line.

Each extremity of the diameter is conjugate to itself; the conjugate of the centre is infinitely distant. The diameter conjugate to the base-diameter is parallel to the lines of reference, and joins the points of contact of the two tangents parallel to the base-diameter.

It will be easy to see what form the equation of the curve assumes when it is referred to conjugate basics. (It is always to be understood, unless otherwise stated, that the direction of reference is that of the conjugate of the base-diameter.) Let the equation of the curve referred to *any* triangle be

$$a_{11}s_1^2 + 2a_{12}s_1s_2 + a_{22}s_2^2 + 2a_{13}s_1s_3 + 2a_{23}s_2s_3 + a_{33}s_3^2 = 0\,.$$

If we put $s_3 = 0$, we shall obtain an equation which is satisfied by the coordinates of the tangents drawn through 3, and also by all lines passing through the intersections of these tangents with the line 12; for the ratio of s_1 to s_2 is constant for all lines passing through a fixed point on the line 12. Now, if

our triangle is a *polar* triangle, these intersections are the points where the line 12 cuts the curve; they are therefore harmonically situated with respect to the points 1 and 2, and the equation which represents them (viz., the equation $a_{11}s_1^2 + 2a_{12}s_1s_2 + a_{22}s_2^2 = 0$, obtained by putting $s_3 = 0$) must be of the form $s_1^2 - \lambda s_2^2 = 0$; that is, the term containing s_1s_2 can not appear. It is evident, in the same way, that the terms containing s_1s_3 and s_2s_3 can not appear; so that the equation of the curve referred to any polar triangle is of the form $a_1s_1^2 + a_2s_2^2 + a_3s_3^2 = 0$. If, then, in this equation, we replace s_1 by s, s_2 by t, s_3 by 1, we shall have the form of the equation of the curve referred to conjugate basics which may therefore be written $\frac{s^2}{c^2} + \frac{t^2}{d^2} = 1$.* Either c^2 or d^2 may be negative, *i. e.*, either c or d imaginary; they cannot both be negative unless the curve itself is imaginary.

We see that for any value of s there are two equal and opposite values of t; *i. e.*, the two tangents drawn from any point of either line of reference cut the other in two points equidistant from the base. This geometrical property follows immediately from the definition of conjugate basics; and we could have inferred the form of the equation from this property, instead of the converse.

Before going any further, let us see how this simple form of the equation is derived from the most general bipunctual equation of the curve $a_{11}s^2 + 2a_{12}st + a_{22}t^2 + 2a_{13}s + 2a_{23}t + a_{33} = 0$. As tripunctual coordinates are interchangeable with trilinear coordinates, and are replaced by bipunctual coordinates in the same manner as trilinear coordinates are replaced by bilinear coordinates, we can here use at once certain algebraic results obtained for the latter (see Clebsch, p. 82, *et seqq.*) The coordinates of the base-diameter are

$$\sigma = \frac{A_{13}}{A_{33}} = \frac{a_{21}a_{32} - a_{22}a_{31}}{a_{11}a_{22} - a_{12}^2}, \qquad \tau = \frac{A_{23}}{A_{33}} = \frac{a_{31}a_{12} - a_{11}a_{32}}{a_{11}a_{22} - a_{12}^2};$$

and I shall suppose, for the present, that A_{33} is not zero. To transfer the initials to the base-diameter (the initials moving in the direction of reference) we put $s = s' + \sigma$, $t = t' + \tau$, and our equation becomes $a_{11}s'^2 + a_{22}t'^2 + 2a_{12}s't' + \frac{A}{A_{33}} = 0$. (It is supposed *throughout* that A, the determinant of $a_{11} \ldots . a_{33}$ is not zero.)

* In the above reasoning, as in general where the case has admitted it, I have followed the method used in Clebsch's "*Vorlesungen über Geometrie.*"

If we represent by m and n the removes of a straight line from the initials S and T, and by ϑ the ratio of the sine of the angle the line makes with the base to the sine of the angle it makes with the lines of reference, we have $s = -m\vartheta$, $t = -n\vartheta$; also $n - m = L$, where L is the distance between the initials. When the line considered is parallel to the base, $\vartheta = 0$; when it is parallel to the lines of reference, $\vartheta = \infty$.

The condition that a line s, t, should pass through the pole of the line s', t', with reference to the curve $a_{11}s^2 + a_{22}t^2 + 2a_{12}st + \frac{A}{A_{33}} = 0$, is $a_{11}ss' + a_{22}tt' + a_{12}(st' + s't) + \frac{A}{A_{33}} = 0$; when the two lines are parallel to the lines of reference, this equation becomes (replacing s by $m\vartheta$, t by $n\vartheta$, s' by $m'\vartheta$, and t' by $n'\vartheta$, and dividing by ϑ, and then making ϑ infinite)

$$a_{11}mm' + a_{22}nn' + a_{12}(mn' + m'n) = 0, \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

or, substituting $L + m$ for n, $L + m'$ for n',

$$(a_{11} + 2a_{12} + a_{22})\, mm' + (a_{12} + a_{22})(m + m')\, L + a_{22}L^2 = 0.$$

If we transfer the initials to points whose distances from the former initials are m, n, and m', n', respectively, we shall find that (1) is the necessary and sufficient condition of the disappearance of the term containing st. To effect the transformation, we replace s by $\frac{mt - m's}{l}$ and t by $\frac{nt - n's}{l}$ (where $l = m - m' = n - n'$ is the distance between the new initials); and the equation becomes $(a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2)\, s^2 + (a_{11}m^2 + 2a_{12}mn + a_{22}n^2)\, t^2 - 2\,[a_{11}mm' + a_{22}nn' + a_{12}(mn' + m'n)]\, st + \frac{A}{A_{33}}\, l^2 = 0$. (1) expresses the necessary and sufficient condition of the disappearance of the term containing st; when that condition is fulfilled, our equation becomes $(a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2)\, s^2 + (a_{11}m^2 + 2a_{12}mn + a_{22}n^2)\, t^2 + \frac{A}{A_{33}}\, l^2 = 0$, or $\frac{s^2}{c^2} + \frac{t^2}{d^2} = l^2$, where $c^2 = -\frac{A}{A_{33}} \cdot \frac{1}{a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2}$, and $d^2 = -\frac{A}{A_{33}} \cdot \frac{1}{a_{11}m^2 + 2a_{12}mn + a_{22}n^2}$.

In the equation $\frac{s^2}{c^2} + \frac{t^2}{d^2} = l^2$, c^2 and d^2 are real quantities, but not necessarily both positive. For $s = t$, we have $\frac{s^2}{c^2} + \frac{s^2}{d^2} = l^2$, whence $s = \pm\, l\sqrt{\frac{c^2d^2}{c^2 + d^2}}$;

hence the length of the diameter conjugate to the base diameter is $2l\sqrt{\dfrac{c^2d^2}{c^2+d^2}}$. Putting $-\mu\vartheta$ for s and $-\nu\vartheta$ for t in the equation of the curve, it becomes $\dfrac{\mu^2\vartheta^2}{c^2}+\dfrac{\nu^2\vartheta^2}{d^2}=l^2$; when $\vartheta=\infty$, we get $\dfrac{\mu^2}{c^2}+\dfrac{\nu^2}{d^2}=0$; or, putting for ν its value $l+\mu$, $\dfrac{\mu^2}{c^2}+\dfrac{(l+\mu)^2}{d^2}=0$, whence $\mu=l\dfrac{-c^2\pm\sqrt{-c^2d^2}}{c^2+d^2}$. The difference between the two values of μ is the length of the base-diameter, which is therefore $\pm 2l\dfrac{\sqrt{-c^2d^2}}{c^2+d^2}$. Representing by $2a$, $2b$, the lengths of the base diameter and its conjugate, respectively, we have, then, $a^2=-l^2\dfrac{c^2d^2}{(c^2+d^2)^2}$, $b^2=l^2\dfrac{c^2d^2}{c^2+d^2}$; whence $c^2+d^2=-\dfrac{b^2}{a^2}$, $c^2d^2=-\dfrac{b^4}{a^2l^2}$, so that c^2 and d^2 are the roots of the equation $x^2+\dfrac{b^2}{a^2}x-\dfrac{b^4}{a^2l^2}=0$, and their values are $-\dfrac{b^2}{2a^2l}(l\pm\sqrt{l^2+4a^2})$.

When $s=0$, $t=\pm ld$; when $t=0$, $s=\pm lc$; that is, the departure from S of a tangent passing through T is lc, and the departure from T of a tangent passing through S is ld; these distances may be called the initial departures of the curve. We have $l^2c^2+l^2d^2=-\dfrac{b^2}{a^2}l^2$, $l^4c^2d^2=-\dfrac{b^4}{a^2}l^2$. Now a and b, being the lengths of the base-diameter and its conjugate, are unaltered when the initials are moved on the base; hence, whenever the initials are conjugate basics on a given diameter, the sum of the squares of the initial departures is proportional to the square of the distance between the initials, and the product of the initial departures is proportional to the distance between the initials. The initial departures cannot both be real unless the initials are on a diameter which cuts the curve in imaginary points and whose conjugate cuts the curve in real points; for it is only when b^2 is positive and a^2 negative that the expressions for c^2 and d^2, namely $-\dfrac{b^2}{2a^2l}(l\pm\sqrt{l^2+4a^2})$, are both positive.

Let us see how our curves can be classified. We have

1) When $c^2+d^2=0$, a and b are both infinite.
2) When c^2+d^2 is negative, a and b are real and finite.
3) When c^2+d^2 is positive, a is imaginary and b real if c^2 and d^2 are both positive; a real and b imaginary if c^2 and d^2 have opposite signs.

(When $c^2 + d^2 = -1$, a and b are equal in absolute value and both real; when $c^2 + d^2 = +1$, a^2 and b^2 are equal in absolute value, but either a or b is imaginary.)

It would seem, then, that we can divide our curves into three classes:

1) Those in which the base-diameter and its conjugate are both infinite in length. ($c^2 + d^2 = 0$).

2) Those in which they are both terminated in real points ($c^2 + d^2 < 0$).

3) Those in which one of the two diameters is terminated in real points and the other in imaginary points ($c^2 + d^2 > 0$).

But we must see whether these characteristics are preserved when the direction of reference is changed; or in other words, whether, if one pair of conjugate diameters belongs to a certain one of these categories, *every* pair of conjugate diameters of the same curve will belong to the same category. Returning to the expressions which we replaced by c^2 and d^2 (page 159), we have

$c^2 + d^2 = -\dfrac{A}{A_{33}}\left(\dfrac{1}{a_{11}m^2 + 2a_{12}mn + a_{22}n^2} + \dfrac{1}{a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2}\right)$. The quantity in the parenthesis is equal to $\dfrac{a_{11}m^2 + 2a_{12}mn + a_{22}n^2 + a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2}{(a_{11}m^2 + 2a_{12}mn + a_{22}n^2)(a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2)}$.

If we subtract from the denominator of this fraction the square of $a_{11}mm' + a_{22}nn' + a_{12}(mn' + m'n)$, which is equal to zero, and subtract from the numerator $2\,[a_{11}mm' + a_{22}nn' + a_{12}(mn' + m'n)]$, the fraction reduces to

$$\frac{a_{11}(m - m')^2 + 2a_{12}(m - m')(n - n') + a_{22}(n - n')^2}{(a_{11}a_{22} - a_{12}^2)(mn' - m'n)^2},$$

or, since $m - m' = n - n' = l$ and $n - m = n' - m' = L$, to $\dfrac{a_{11} + 2a_{12} + a_{22}}{A_{33}L^2}$; so that we have

$$c^2 + d^2 = -\frac{A}{A_{33}^2L^2}(a_{11} + 2a_{12} + a_{22}).$$

It will be convenient to postpone the investigation of the change which this expression undergoes when the direction of reference is changed; but I must anticipate the results of that investigation so far as to say that A becomes $\alpha^2 A$, where α is a real trigonometrical function which cannot vanish; and that $a_{11} + 2a_{12} + a_{22}$ remains entirely unchanged. So we see—since the denominator is essentially positive—that the sign of the above expression is unal-

tered by a change in the direction of reference; and that if it is zero for any direction of reference, it is zero for all. We have, then, the general classification:

1) $a_{11}+2a_{12}+a_{22}$ is zero; all diameters have infinite length (parabola).

2) $a_{11}+2a_{12}+a_{22}$ has the same sign as A; all diameters have finite length (ellipse).

3) $a_{11}+2a_{12}+a_{22}$ has the sign contrary to that of A; of each pair of conjugate diameters, one has real length and one imaginary length (hyperbola).

We have found $c^2+d^2=-\dfrac{A}{A_{33}^2L^2}(a_{11}+2a_{12}+a_{22})$, and we have also $c^2d^2=\dfrac{A^2}{A_{33}^2}\cdot\dfrac{1}{(a_{11}m^2+2a_{12}mn+a_{22}n^2)(a_{11}m'^2+2a_{12}m'n'+a_{22}n'^2)}=\dfrac{A^2}{A_{33}^3L^2l^2}$. For any given direction of reference, all the quantities involved in these values, except l, are constant; so that we see again that $(c^2+d^2)\,l^2$ is proportional to l^2 and that cdl^2 is proportional to l.

Let us investigate the condition of the equality of conjugate diameters. That condition is expressed (see p. 161, l. 1) by the equation $-\dfrac{A}{A_{33}^2L^2}(a_{11}+2a_{12}+a_{22})=-1$. If we represent by ϕ the angle made by the *original* base with the lines of reference, and call the distance between the original initials unity, we have, since $\dfrac{A_{13}}{A_{33}}$ and $\dfrac{A_{23}}{A_{33}}$ are the coordinates of the base-diameter, $L^2=1+\left(\dfrac{A_{13}-A_{23}}{A_{33}}\right)^2+2\,\dfrac{A_{13}-A_{23}}{A_{33}}\cos\phi$, and the equation of condition becomes $\dfrac{A\,(a_{11}+2a_{12}+a_{22})}{A_{33}^2+(A_{13}-A_{23})^2+2A_{33}(A_{13}-A_{23})\cos\phi}=1$.—Let the equation of the curve referred to a rectangular system be

$$a_{11}s^2+2a_{12}st+a_{22}t^2+2a_{13}s+2a_{23}t+a_{33}=0.$$

Let the direction of reference be altered so that the lines of reference shall make an angle ω' with the base; denote the coefficients of the new equation by a'_{11}, &c., their determinant by A', and its minors by A'_{11}, &c. The transformation is effected by means of the formulæ (p. 153) $s=\dfrac{s'}{\alpha+\beta\,(s'-t')}$,

$t = \dfrac{t'}{\alpha + \beta\,(s' - t')}$; where $\alpha = \dfrac{1}{\sin\omega'}$ and $\beta = \dfrac{\cos\omega'}{\sin\omega'}$, so that $\cos\omega' = \dfrac{\beta}{\alpha}$ and $\alpha^2 = 1 + \beta^2$. We have

$$a'_{11} = a_{11} + 2a_{13}\beta + a_{33}\beta^2\,, \qquad a'_{22} = a_{22} - 2a_{23}\beta + a_{33}\beta^2\,, \qquad a'_{33} = a_{33}\alpha^2\,,$$
$$a'_{12} = a_{12} - (a_{13} - a_{23})\,\beta - a_{33}\beta^2\,, \qquad a'_{13} = a_{13}\alpha + a_{33}\alpha\beta\,, \qquad a'_{23} = a_{23}\alpha - a_{33}\alpha\beta\,;$$

whence we find

$$A' = \begin{vmatrix} a_{11} + 2a_{13}\beta + a_{33}\beta^2, & a_{12} - (a_{13} - a_{23})\beta - a_{33}\beta^2, & a_{13}\alpha + a_{33}\alpha\beta \\ a_{12} - (a_{13} - a_{23})\beta - a_{33}\beta^2, & a_{22} - 2a_{23}\beta + a_{33}\beta^2, & a_{23}\alpha - a_{33}\alpha\beta \\ a_{13}\alpha + 0 + a_{33}\alpha\beta, & a_{23}\alpha + 0 - a_{33}\alpha\beta, & 0 + a_{33}\alpha^2 \end{vmatrix} = \alpha^2 A,$$

$$A'_{33} = A_{33} - 2\beta\,(A_{13} - A_{23}) + \beta^2\,(A_{11} + A_{22} - 2A_{12}) = A_{33} - 2\beta G + \beta^2 H,$$
$$A'_{13} - A'_{23} = -\alpha\beta\,(A_{11} + A_{22} - 2A_{12}) + \alpha\,(A_{13} - A_{23}) = \alpha G - \alpha\beta H,$$

(where, for brevity, we put $A_{13} - A_{23} = G$, $A_{11} + A_{22} - 2A_{12} = H$) and $a'_{11} + 2a'_{12} + a'_{22} = a_{11} + 2a_{12} + a_{22}$. If, now, in the equation

$$\frac{A'\,(a'_{11} + 2a'_{12} + a'_{22})}{A'^2_{33} + (A'_{13} - A'_{23}) + 2A'_{33}\,(A'_{13} - A'_{23})\cos\phi} = 1\,,$$

we substitute the values above obtained, and substitute $\dfrac{\beta}{\alpha}$ $(= \cos\omega')$ for $\cos\phi$, we obtain, after reduction,

$$A^2_{33} + G^2 - A\,(a_{11} + 2a_{12} + a_{22}) - 2G\,(A_{33} + H)\,\beta$$
$$+ \left[G^2 + H^2 - A\,(a_{11} + 2a_{12} + a_{22})\right]\beta^2 = 0.$$

From this equation we can, in general, obtain two values of β, which give us the directions of the two equal conjugate diameters.

In order that *every* diameter should be equal to its conjugate, the above equation must be satisfied for all values of β, and we must have

$$A^2_{33} + G^2 - A\,(a_{11} + 2a_{12} + a_{22}) = 0, \quad G\,(A_{33} + H) = 0,$$
$$\text{and } G^2 + H^2 - A\,(a_{11} + 2a_{12} + a_{22}) = 0.$$

Substituting for G its value, and remembering that $Aa_{11} = A_{22}A_{33} - A^2_{23}$, &c., the first equation becomes $2\,(A_{13} - A_{23})^2 - A_{33}\,(A_{11} - 2A_{12} + A_{22} - A_{33}) = 0$, or $2G^2 - A_{33}(H - A_{33}) = 0$, and the third becomes likewise $2G^2 - A_{33}H + H^2 = 0$, or $2G^2 + H\,(H - A_{33}) = 0$. We have, then,

$$(1)\;\ldots\;2G^2 - A_{33}\,(H - A_{33}) = 0\,; \qquad (2)\;\ldots\;G\,(H + A_{33}) = 0\,;$$
$$(3)\;\ldots\;2G^2 + H\,(H - A_{33}) = 0.$$

Equation (2) shows that we must have either $G = 0$ or $H + A_{33} = 0$. In the first case, we must also have $H - A_{33} = 0$ in order that equations (1) and

(3) should be satisfied; in the second case, equations (1) and (3) become $G^2 + A_{33}^2 = 0$, which cannot be satisfied if the coefficients of the given equation are real, as we suppose them to be throughout. The necessary and sufficient condition, therefore, that every diameter should be equal to its conjugate is, in any rectangular system of coordinates, (the distance between the initials being taking as unity)

$$G = 0 \text{ and } H - A_{33} = 0; \text{ that is, } A_{13} = A_{23} \text{ and } A_{11} - 2A_{12} + A_{22} = A_{33}.$$

The square of half the diameter making the angle $\cot^{-1}\beta$ with the original base is (pp. 160, 162)

$$l^2 \frac{c^2d^2}{c^2+d^2} = -\frac{A'}{A'_{33}(a'_{11}+2a'_{12}+a'_{22})} = -\frac{\alpha^2 A}{(A_{33}-2\beta G+\beta^2 H)(a_{11}+2a_{12}+a_{22})};$$

when $G = 0$ and $H = A_{33}$, this becomes

$$-\frac{\alpha^2 A}{(1+\beta^2) A_{33}(a_{11}+2a_{12}+a_{22})} = -\frac{A}{A_{33}(a_{11}+2a_{12}+a_{22})},$$

an expression independent of β. So we see that when every diameter is equal to its conjugate, all diameters are equal; *i. e.*, the curve is a circle. The condition $A_{13} = A_{23}$ shows that the base-diameter is parallel to the original base; we have seen that this condition holds in the case of the circle for *any* rectangular system: therefore, in the circle, every diameter is perpendicular to its conjugate.

The condition $c^2 + d^2 = +1$ (see p. 161, l. 2) would become, in the same way,

$$-(1+\beta^2) A (a_{11}+2a_{12}+a_{22}) = A_{33}^2 + G^2 - 2G(A_{33}+H)\beta + (G^2+H^2)\beta^2,$$

or

$$A_{33}^2 + G^2 + A(a_{11}+2a_{12}+a_{22}) - 2G(A_{33}+H)\beta$$
$$+ [G^2 + H^2 + A(a_{11}+2a_{12}+a_{22})]\beta^2 = 0;$$

and we should find, in like manner, that if this equation is satisfied for all values of β we have $A_{33}(H+A_{33}) = 0$, $G(H+A_{33}) = 0$, and $H(H+A_{33}) = 0$, so that the only condition here is $H + A_{33} = 0$.

It may be worth while to observe that the results obtained for A', A'_{33}, $A'_{13} - A'_{23}$, and $a'_{11} + 2a'_{12} + a'_{22}$, are entirely independent of the fact that the original system was rectangular; they apply to all cases: and it is only in the replacing of α^2 by $1 + \beta^2$ and of $\cos\phi$ by $\frac{\beta}{\alpha}$, that the rectangularity of the original system comes into play, and simplifies the results.

The angle between the base-diameter and its conjugate (Fig. 8) is given by

$$\frac{\sin\delta}{\sin(\omega'-\delta)}=\frac{1}{\frac{A'_{13}}{A'_{33}}-\frac{A'_{23}}{A'_{33}}}=\frac{A'_{33}}{A'_{13}-A'_{23}};\text{ whence }\tan\delta=\frac{A'_{33}\sin\omega'}{A'_{13}-A'_{23}+A'_{33}\cos\omega'}$$

$$=\frac{A'_{33}}{(A'_{13}-A'_{23})\,\alpha+A'_{33}\beta}=\frac{A_{33}-2\beta G+\beta^2H}{\alpha^2G-\alpha^2\beta H+\beta A_{33}-2\beta^2G+\beta^3H}$$

$$=\frac{A_{33}-2\beta G+\beta^2H}{(1-\beta^2)\,G-\beta H+\beta A_{33}};\text{ so that we have the equation}$$

$$\beta^2(H+G\tan\delta)+\beta\,[(H-A_{33})\tan\delta-2G]-G\tan\delta+A_{33}=0\,.$$

The condition for the reality of the values of β is

$$4\,(A_{33}-G\tan\delta)(H+G\tan\delta)\;\overline{\overline{<}}\;[(H-A_{33})\tan\delta-2G]^2\,,$$

which reduces to

$$\tan^2\delta\;\overline{\overline{>}}\;4\,\frac{A_{33}H-G^2}{(H-A_{33})^2+4G}\,.$$

This condition is satisfied for all values of δ if $A_{33}H-G^2$ is not positive, *i. e.* if $A_{33}(A_{11}-2A_{12}+A_{22})-(A_{13}-A_{23})^2\;\overline{\overline{<}}\;0$, and this can be put into the form $A_{11}A_{33}-A_{13}^2+A_{22}A_{33}-A_{23}^2+2\,(A_{13}A_{23}-A_{12}A_{33})\;\overline{\overline{<}}\;0$, or $A\,(a_{11}+2a_{12}+a_{22})\;\overline{\overline{<}}\;0$. Thus we see that in the parabola and hyperbola the conjugate diameters may make any angle with each other, but that in the ellipse there is a minimum angle given by the equation $\tan^2\delta=4\,\frac{A_{33}H-G^2}{(H-A_{33})^2+4G^2}$. In the case of the circle, $H-A_{33}=0$ and $G=0$, and this equation becomes $\tan^2\delta=\infty$, whence we see again that in the circle every diameter is perpendicular to its conjugate.

Solving the quadratic in β found above, we obtain

$$\beta=\frac{-(H-A_{33})\tan\delta+2G\pm\sqrt{[(H-A_{33})^2+4G^2]\tan^2\delta-4\,(A_{33}H-G^2)}}{2\,(H+G\tan\delta)}\,.$$

Since $A_{33}H-G^2\equiv A\,(a_{11}+2a_{12}+a_{22})$, we have, in the case of the parabola, $A_{33}H-G^2=0$, and the values of β will be found to reduce to

$$\beta=\frac{A_{33}-G\tan\delta}{G+A_{33}\tan\delta}\text{ and }\beta=\frac{A_{33}}{G}\,.$$

The second of these values is independent of δ; therefore in the parabola there is a fixed direction in which one diameter in every pair runs. In order that this direction should coincide with the direction of reference we must have

$\beta = 0$, whence $A_{33} = 0$. Now, when $A_{33} = 0$, the coordinates of the base-diameter are infinite; and since we have seen that the conjugate diameters may make any angle with each other, this cannot be interpreted to mean that the base-diameter is parallel to the lines of reference, *i. e.* to its conjugate: it must, then, be infinitely distant, and we have therefore found that in the parabola all diameters that are not infinitely distant are parallel to each other.

Let the equation of a conic referred to a certain pair of conjugate basics be $\frac{s^2}{C^2} + \frac{t^2}{D^2} = L^2$, and its equation, referred to any other pair of conjugate basics on the same diameter, $\frac{s^2}{c^2} + \frac{t^2}{d^2} = l^2$. Putting $C^2 + D^2 = g$, we know that, wherever the new basics be taken, we have $c^2 + d^2 = g$ and $l^2c^2d^2 = L^2C^2D^2$; and if we assign any particular value to l, c, or d, we can obtain the values of the other two of these quantities from these equations. If we put $d^2 = -1$, so that the equation of the curve becomes $s^2 = c^2(t^2 + l^2)$, we have $c^2 = g + 1$, $l^2 = -\frac{C^2D^2}{g+1}L^2$. The condition that the basics for which the equation of the curve assumes this form should be real points is, that this value of l^2 should be positive; and it is easy to see when this condition is fulfilled:

In the parabola, C^2D^2 is negative and $g = 0$; hence l^2 is positive.

In the ellipse, C^2D^2 is negative and g is negative; so that l^2 is positive when g is numerically less than 1, and negative when g is numerically greater than 1; that is (p. 160), positive on a diameter that is greater than its conjugate and negative on one that is smaller than its conjugate.

In the hyperbola g is always positive; C^2D^2 is negative when the diameter cuts the curve, and positive when it does not; in the former case l^2 is positive, in the latter, negative.

Of course, to every numerical value of l there correspond two pairs of conjugate basics, symmetrically situated with respect to the centre; in the parabola, one of these pairs is infinitely distant.

It appears from the discussion of the angles made by diameters with their conjugates, that every conic has one pair of rectangular conjugate diameters, or axes; and we now see that on *one* of the axes there can be found, on each side of the centre, a pair of conjugate basics for which the equation of the curve assumes the form $s^2 = e^2(t^2 + l^2)$; that is, a pair of

conjugate basics, S and T, such that *the ratio of the distances from S to the intersections of any tangent with the S and T lines is constant.* It is plain that e^2 is always positive, and that in the parabola $e^2 = 1$, in the ellipse $e^2 < 1$, and in the hyperbola $e^2 > 1$.

Let us see how, by means of this property, we can construct the tangents to a conic from a given point

For the parabola, the construction is very simple. Let S and T (Fig. 9) be the basics in question, and M the given point. The line drawn from S to the point midway between the intersections of the tangent with the S and T lines is perpendicular to the tangent (since S is equidistant from those intersections); that point is consequently on the circle described upon SM as a diameter. But it is also on the line drawn parallel to the S and T lines midway between them; it is therefore at the intersection of this line with the above-mentioned circle. Hence, to construct the tangents from M, we draw OP midway between the S and T lines; and from N, the middle point of MS, we describe a circle with NS as radius; the lines joining M with the intersections of this circle with OP are the required tangents. If $NO = NS$, the tangents coincide and M is a point of the parabola; and conversely, if M is a point of the parabola, the tangents coincide and $NO = NS$. But $\frac{NO}{NS} = \frac{MQ}{MS}$; therefore, the distance of any point of the parabola from S is equal to its distance from the T line. When MQ is greater than MS, the two tangents are imaginary; when MQ is less than MS, the two tangents are real and distinct; when M is on the T line, the two tangents are perpendicular to each other. We may call S the focus and the T line the directrix of the parabola.

Let us proceed to the ellipse and hyperbola. We have, for any tangent MO (Fig. 10), $\frac{SN}{SO} = e$. Denote the distance ST by Ef, where $E = \frac{1}{e}$, take $SP = f$, and draw PQ parallel to the lines of reference; it intersects SO at a point Q. It is obvious that $SQ = SN$, and QN is a tangent to the parabola whose focus is S and directrix PQ. If, therefore, we can determine the point U where QN intersects SM, we can construct the tangents from M by constructing the tangents from U to the parabola (SP) and joining M with the intersections of those tangents with the S line. Denote the distance SM by Ez and take $SR = z$. RQ is parallel to MO; hence, if we draw *any* line MV, meeting the S line at V, and a parallel line RW meeting PQ at W, U is the

intersection of VW with SM. We have, then, the following construction for the tangents to the ellipse (or hyperbola) from the point M, the curve being given by S, T, and P. Through M draw any line meeting the S line at V; on SM take a point R such that $\frac{SR}{SM} = \frac{SP}{ST}$, and through R draw a line parallel to MV, meeting the P line at W; join VW, and from U, the intersection of VW with SM, construct the tangents to the parabola (SP); the lines joining M with the points where these tangents cut the S line are the required tangents.

Let us take MV and RW *parallel to* ST, and produce MV to meet the T line in X; also draw UY parallel to ST and meeting the P line in Y. It can be shown* that $\frac{US}{UY} = E\,\frac{MS}{MX}$; whence it follows at once (from what we have found for the parabola) that the distance of any point on the ellipse (or hyperbola) from the focus is equal to e times its distance from the directrix—S being again designated the focus and the T line the directrix; also that no tangent can be drawn when the ratio of these distances is less than e, and two distinct tangents when it is greater than e.

It is plain that in the ellipse the foci lie between the extremities of the axis and the directrices outside of them, and that the reverse is the case with the hyperbola; it follows immediately from the above, therefore, that in the ellipse the sum of the distances of any point from the two foci is constant, and in the hyperbola the difference of the distances of any point from the two foci is constant.

The following are obvious consequences of the property by which the focus and directrix were brought to our notice: in the parabola, any tangent makes, at its intersection with the directrix, equal angles with the directrix and the line joining that intersection with the focus; in the ellipse and hyperbola, any tangent makes, at its intersections with the two directrices, equal angles with the lines joining those intersections with the corresponding foci.

---

* Denote MX by Ey; then $RW = y$. We have

$$RU = RM\,\frac{RW}{RW - MV} = (E-1)\,z\,\frac{y}{Ef - (E-1)\,y}, \qquad SU = SR + RU = z + (E-1)\,z\,\frac{y}{Ef - (E-1)\,y}$$

$$= \frac{Efz}{Ef - (E-1)\,y}; \qquad UZ = SU\,\frac{MV}{SM} = \frac{Efz}{Ef - (E-1)\,y}\cdot\frac{E\,(y-f)}{Ez} = \frac{Ef\,(y-f)}{Ef - (E-1)\,y}, \qquad UY = UZ + f$$

$$= \frac{fy}{Ef - (E-1)\,y}; \quad \text{therefore } \frac{SU}{UY} = \frac{Ez}{y} = E\,\frac{Ez}{Ey} = E\,\frac{SM}{MX}.$$

Let $\frac{s^2}{C^2} + \frac{t^2}{D^2} = L^2$ be the equation of a conic referred to any pair of conjugate basics on an axis; is there a point on the axis the ratio of whose distances from the intersections of any tangent with the S and T lines is constant? Writing the equation in the form $s^2 = -\frac{C^2}{D^2}t^2 + C^2L^2$, we see that if such a point exists, then, denoting its distance from S by αL (whence its distance from T is $(1+\alpha)\,L$), the above equation must be identical with $s^2 + \alpha^2L^2 = -\frac{C^2}{D^2}\left[t^2 + (1+\alpha)^2\,L^2\right]$; so that we have, for determining α, $\frac{C^2}{D^2}(1+\alpha)^2 + \alpha^2 = -C^2$, whence

$$\alpha = \frac{-C^2 \pm \sqrt{-C^2D^2\,(C^2+D^2+1)}}{C^2+D^2} = \frac{-C^2 \pm \sqrt{-C^2D^2\,(g+1)}}{g}.$$

It is plain that the condition of the reality of these values is the same as that of the existence of real foci (see p. 166, l. 14); so that points possessing the property in question can be found only on the axis on which the foci are situated. The distance from S of the point midway between the two points corresponding to the two values of α is $\frac{\alpha_1+\alpha_2}{2}L = -\frac{C^2}{g}L$, which, it will be readily seen from p. 160, l. 4, is the distance of the centre; so that the two points are equidistant from the centre. The distance between the points is

$$(\alpha_1 - \alpha_2)\,L = \frac{2\sqrt{-C^2D^2L^2\,(g+1)}}{g} = \frac{2\sqrt{e^2l^2\,(g+1)}}{g} = \frac{2e^2l}{g}\ (\text{since } g+1 = e^2);$$

and this is obviously equal, in absolute value, to twice the distance from the focus to the centre; that is, equal to the distance between the foci. Hence, the points sought are the foci themselves, and we have the following theorem (in which the term "conjugate ordinates" is to be understood as meaning ordinates to the transverse axis passing through conjugate basics on that axis):

If M and M' be the points in which *any* tangent to a conic intersects a *fixed* pair of conjugate ordinates, and F a focus of the conic, the ratio $\frac{FM}{FM'}$ is constant, and the same for both foci.*

* This theorem admits of several easy geometrical demonstrations.

It may be remarked that this ratio, $\frac{C^2}{D^2}$, is unity for all pairs of conjugate ordinates in the parabola; in the ellipse and hyperbola it varies for different pairs, from 0 to 1 or from 1 to ∞.

The following are one or two obvious consequences of this proposition:

1. Since the ordinate conjugate to the minor axis is a straight line perpendicular to the transverse axis and infinitely distant, and since the distance from the focus to the intersection of any tangent with this infinitely distant ordinate is proportional to the secant of the angle made by the tangent with the transverse axis; it follows from the theorem just established that the distance from the focus to the intersection of any tangent with the minor axis is proportional to the secant of the angle made by the tangent with the transverse axis. In the case of the ellipse, this distance is equal to half the transverse axis when the angle is 0; so that the distance from the focus to the intersection of any tangent with the minor axis is equal to the distance intercepted by the axes on a line drawn through an extremity of the transverse axis, parallel to the tangent. It can easily be seen that this result is true of the hyperbola also, within the limits to which its real tangents are confined.

2. Let M be the point of contact of a tangent, and M' the point at which it intersects the transverse axis. We have, by the above theorem, $\frac{FM}{FM'} = \frac{F'M}{F'M'}$, whence we see that the angle $F'MM'$ is the supplement of FMM'; that is, the tangent makes equal angles with the lines joining its point of contact with the foci.

Finally, let us ascertain whether the foci or any other points enjoy a property like that which we have been considering, with respect to any pair of parallel lines other than conjugate ordinates. Refer the conic to any rectangular system with the given parallels as lines of reference. Denote by a the distance from the base of a point possessing the property in question, by b its distance from the S line, by L the distance between the parallels, and by m an indeterminate constant; then the equation of the conic is

$$(s - a)^2 + b^2 = m\,[(t - a)^2 + (b + L)^2], \text{ or}$$

$$s^2 - mt^2 - 2as + 2mat + a^2 + b^2 - m\,[a^2 + (b + L)^2] = 0.$$

Hence we have, with the usual meanings for A_{13} &c.,

$$A_{13} = -ma,\ A_{23} = -ma,\ A_{33} = -m,\ \frac{A_{13}}{A_{33}} = \frac{A_{23}}{A_{33}} = a;$$

and it follows at once (see p. 158, l. 26) that the parallels are perpendicular to an axis and that the point or points sought lie on the axis; and the absence of the term containing st shows that the parallels cut the axis in conjugate basics. We are thus brought back to the case already investigated; and conjugate ordinates are therefore the only pairs of parallels, and the foci the only points, enjoying the property in question. That it cannot belong to lines not parallel is evident.

The Conic referred to its Foci.—The distances of the extremities of the axis from the S initial are (p. 160) $l\dfrac{-c^2 \pm \sqrt{-c^2d^2}}{c^2+d^2}$, so that the distance of the centre from the S initial is $\dfrac{-lc^2}{c^2+d^2}$; and in like manner the distance of the centre from the T initial is $\dfrac{ld^2}{c^2+d^2}$. When the S initial is a focus, these distances are $\dfrac{e^2l}{1-e^2}$, $\dfrac{l}{1-e^2}$; so that the distances of the other focus from S and T are $\dfrac{2e^2l}{1-e^2}$ and $\dfrac{1+e^2}{1-e^2}l$. To transfer (see p. 152) the initials to the foci, therefore, we leave s unchanged and replace t by $\dfrac{lt-\dfrac{1+e^2}{1-e^2}ls}{\dfrac{2e^2l}{1-e^2}} = \dfrac{(1-e^2)t-(1+e^2)s}{2e^2}$,

and the equation $s^2 = e^2(t^2+l^2)$ becomes $4e^2s^2 = (1-e^2)^2t^2 + (1+e^2)^2s^2 - 2(1-e^4)st + 4e^4l^2$ or $s^2 + t^2 - 2\dfrac{1+e^2}{1-e^2}st + \dfrac{4e^4l^2}{(1-e^2)^2} = 0$. The last term in this equation is the square of the distance between the foci; denoting that distance by $2k$, and $\dfrac{1+e^2}{1-e^2}$ by h, the equation becomes

$$s^2 + t^2 - 2hst + 4k^2 = 0.$$

The equation of any other conic having the same foci would be $s^2 + t^2 - 2h'st + 4k^2 = 0$; combining this with the preceding equation, we have $st = 0$ for the common tangents of the two curves. This condition is satisfied only by lines passing through the foci; and the equation of the curve shows that tangents passing through the foci are imaginary. Hence, all confocal conics have in common four imaginary tangents passing through the foci.

Denoting by σ, τ, the lengths of the perpendiculars dropped from S, T, upon the line s, t, and by ρ the cosine of the angle made by the line with the

base, we have $s = \dfrac{\sigma}{\rho}$, $t = \dfrac{\tau}{\rho}$, $\rho^2 = \dfrac{L^2 - (\sigma - \tau)^2}{L^2}$. By means of these equations we can transform any equation in s, t, in a rectangular system, into an equation in σ, τ; and the equation of the conic referred to its foci assumes, when thus transformed, a very simple form. We have, first, $\sigma^2 + \tau^2 - 2h\sigma\tau + 4k^2\rho^2 = 0$; but $\rho^2 = \dfrac{4k^2 - (\sigma - \tau)^2}{4k^2}$, and the equation reduces to $2(1 - h)\,\sigma\tau + 4k^2 = 0$, or $\sigma\tau = \dfrac{2k^2}{h - 1} = \dfrac{e^2 l^2}{1 - e^2} = b^2$, where b is half the minor axis (see p. 160, l. 8). Hence, the product of the perpendiculars dropped from the foci upon any tangent is equal to the square of half the minor axis.

To remove T to the centre, we have simply to replace t by $2t - s$ in the equation $s^2 + t^2 - 2hst + 4k^2 = 0$; so that we have for the equation of the curve referred to a focus and the centre, $\dfrac{1+h}{2} s^2 + t^2 - (1 + h)\, st + k^2 = 0$; from which we obtain

$$\frac{1+h}{2}\sigma^2 + \tau^2 - (1 + h)\,\sigma\tau + k^2\rho^2 = 0, \text{ or } \tau^2 + k^2\rho^2 = \frac{1+h}{2}\sigma\,(2\tau - \sigma).$$

Now $\tau^2 + k^2\rho^2$ is the square of the distance from the centre to the foot of the perpendicular dropped from the focus, and $\sigma\,(2\tau - \sigma)$ is the product of the perpendiculars from the two foci, which is constant and equal to b^2: hence the distance from the centre to the foot of the perpendicular dropped from the focus upon any tangent is constant and equal to $\sqrt{\dfrac{1+h}{2} b^2} = \sqrt{\dfrac{1}{1 - e^2} b^2} = a$; *i. e.*, the feet of all these perpendiculars are on a circle described upon the major axis as diameter.

The Conic referred to the Extremities of a Diameter. — Taking up the general equation of a conic referred to conjugate basics, $\dfrac{s^2}{c^2} + \dfrac{t^2}{d^2} = l^2$, and making the general substitution for a motion of the initials along the base; $s = \dfrac{mt' - m's'}{\lambda}$, $t = \dfrac{nt' - n's'}{\lambda}$ (where λ is the distance between the new initials), we obtain, dropping the accents of the variables,

$$\left(\frac{m'^2}{c^2} + \frac{n'^2}{d^2}\right) s^2 + \left(\frac{m^2}{c^2} + \frac{n^2}{d^2}\right) t^2 - 2\left(\frac{mm'}{c^2} + \frac{nn'}{d^2}\right) st = l^2\lambda^2.$$

If the extremities of the diameter are taken as the new initials, we have (p. 160) $\frac{m'^2}{c^2} + \frac{n'^2}{d^2} = 0$, $\frac{m^2}{c^2} + \frac{n^2}{d^2} = 0$; also, (since m, m', are the roots of the equation $(c^2 + d^2)\mu^2 + 2c^2l\mu + c^2l^2 = 0$, and n, n', the roots of the equation $(c^2 + d^2)\nu^2 - 2d^2l\nu + d^2l^2 = 0$) $mm' = \frac{c^2l^2}{c^2 + d^2}$, $nn' = \frac{d^2l^2}{c^2 + d^2}$, so that the equation of the curve referred to the extremities of a diameter is

$$-\frac{4st}{c^2 + d^2} = \lambda^2 = 4a^2; \text{ or } \left(\text{since } c^2 + d^2 = -\frac{b^2}{a^2}\right), \; st = b^2;$$

that is, the product of the distances cut off by any tangent on the tangents at the extremities of a diameter is equal to the square of half the conjugate diameter.

The equation of a conic in point-coordinates has for its coefficients the minors of the determinant of the coefficients of its equation in line-coordinates; hence the equation in point-coordinates of the conic referred to the extremities of a diameter is $b^2pq - \frac{1}{4} = 0$, or $pq = \frac{1}{4b^2}$. Representing, then, by π, $\varkappa$, the negative reciprocals of p, q, we have $\pi\varkappa = 4b^2$; that is, the lines joining any point of the conic with the extremities of a diameter cut off, on the tangents at those extremities, distances whose product is equal to the square of the conjugate diameter.